Solution to Assignment 2

Supplementary Problems

Note the notations. These problems are valid in all dimensions. Hence we do not use (x, y) to denote a generic point as we do in $\mathbb{R}^2$. Instead, here $\mathbf{x}$ or $\mathbf{p}$ are used to denote a generic point in $\mathbb{R}^n$.

1. Let S be a non-empty set in $\mathbb{R}^n$. Define its characteristic function χ_S to be $\chi_S(\mathbf{x}) = 1$ for $\mathbf{x} \in S$ and $\chi_S(\mathbf{x}) = 0$ otherwise. Prove the following identities:

   (a) $\chi_{A\cup B} = \chi_A + \chi_B - \chi_{A\cap B}$.

   (b) $\chi_{A\cap B} = \chi_A \chi_B$.

   Solution. (a) Exhaust all possible cases (1) $x \in A$ but not in B, (2) $x \in B$ but not in A (3) $x \in A \cap B$, and (4) x does not belong to A nor to B. In all these cases, the identity holds.

   (b) When $x \in A \cap B$, both $\chi_A(x)$ and $\chi_B(x)$ are equal to 1, hence their product is equal to 1. When x does not belong to A or B, one of $\chi_A(x)$ and $\chi_B(x)$ must be 0, hence the product becomes 0.

2. Let f and g be continuous on the region D. Deduce the inequality

$$2 \iint_D |fg|\, dA \le \iint_D f^2\, dA + \iint_D g^2\, dA \ .$$

   Hint: Use $(a \pm b)^2 \ge 0$.

   Solution. We have $(f(x, y) \pm g(x, y))^2 \ge 0$, that is, $f^2(x, y) + g^2(x, y) \ge 2|f(x, y)g(x, y)|$. Integrating this inequality over D to get

$$\iint_D f^2\, dA + \iint_D g^2\, dA \ge 2 \iint_D |fg|\, dA \ .$$

   Note that we have used linearity and positivity of the Riemann integral.

3. Let f be a non-negative continuous function on D and p a positive number. Show that

$$m \le \left(\frac{1}{|D|} \iint_D f^p\, dA \right)^{1/p} \le M \ ,$$

   where m and M are respectively the minimum and maximum of f and $|D|$ is the area of D.

   Solution. From $m^p \le f(x, y)^p \le M^p$, we integrate to get

$$m^p|D| = \iint_D m^p\, dA \le \iint_D f^p\, dA \le= \iint_D M^p\, dA = M^p|D| \ ,$$

   and the inequality follows.

   Note. It shows $\left(|D|^{-1} \iint_D f^p\, dA\right)^{1/p}$ can also be used to describe some kind of average. The cases $p = 1, 2$ are most common.